

FREE ERGODIC \mathbb{Z}^2 -SYSTEMS AND COMPLEXITY

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ABSTRACT. Using results relating the complexity of a two dimensional subshift to its periodicity, we obtain an application to the well-known conjecture of Furstenberg on a Borel probability measure on $[0, 1)$ which is invariant under both $x \mapsto px \pmod{1}$ and $x \mapsto qx \pmod{1}$, showing that any potential counterexample has a nontrivial lower bound on its complexity.

1. INTRODUCTION

1.1. Complexity and periodicity. For a one dimensional symbolic system (X, σ) , meaning that $X \subset \mathcal{A}^{\mathbb{Z}}$ is a closed set, where \mathcal{A} is a finite alphabet, that is closed under the left shift $\sigma: \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$, the Morse-Hedlund Theorem gives a simple relation between the complexity of the system and periodicity. Namely, if $P_X(n)$ denotes the complexity function, which counts the number of nonempty cylinder sets of length n in X , then (X, σ) is periodic if and only if there exists $n \in \mathbb{N}$ such that $P_X(n) \leq n$. Both periodicity and complexity have natural generalizations to higher dimensional systems. For example, for a two dimensional system (X, σ, τ) , meaning that $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a closed set that is invariant under the left and down shifts $\sigma, \tau: \mathcal{A}^{\mathbb{Z}^2} \rightarrow \mathcal{A}^{\mathbb{Z}^2}$, the two dimensional complexity $P_X(n, k)$ is the number of nonempty n by k cylinder sets. In a partial solution to Nivat's Conjecture [11], the authors [2] showed that if (X, σ, τ) is a transitive \mathbb{Z}^2 -subshift and there exist $n, k \in \mathbb{N}$ such that $P_X(n, k) \leq nk/2$, then there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $\sigma^i \tau^j x = x$ for all $x \in X$. In this note, we give an application of this theorem to Furstenberg's well-known " $\times p, \times q$ problem."

1.2. The $\times p, \times q$ problem. Let $S, T: [0, 1) \rightarrow [0, 1)$ denote the maps $Sx := px \pmod{1}$ and $Tx := qx \pmod{1}$, where $p, q \geq 1$ are multiplicatively independent integers (meaning that p and q are not both powers of the same integer). In the 1960's, Furstenberg [4] proved that any closed subset of $[0, 1)$ that is invariant under both S and T is either all of $[0, 1)$ or is finite. He asked whether a similar statement holds for measures:

Conjecture 1.1 (Furstenberg). *Let μ be a Borel probability measure on $[0, 1)$ that is invariant under both S and T and is ergodic for the joint action of S and T . Then either μ is Lebesgue measure or μ is atomic.*

Progress was made in the 1980's with the work of Lyons [9], followed soon thereafter by Rudolph's proof that positive entropy $h_\mu(\cdot)$ of the measure μ with respect to one of the transformations implies the result for relatively prime p and q . This was generalized to multiplicatively independent integers by Johnson:

The second author was partially supported by NSF grant 1500670.

Theorem 1.2 (Rudolph [16] and Johnson [6]). *Let μ be a Borel probability measure on $[0, 1)$ that is invariant under both S and T and is ergodic for the joint action of S and T . If $h_\mu(S) > 0$ (or equivalently $h_\mu(T) > 0$), then μ is Lebesgue measure.*

One way to interpret this theorem is that the set of $\langle S, T \rangle$ -ergodic measures experiences an entropy gap with respect to the one-dimensional action generated by S (or equivalently by T). Informally, if μ has *high entropy* (in this case meaning that $h_\mu(S) > 0$), then its entropy with respect to S is actually $\log p$ and μ is Lebesgue measure. Our main theorem is that the set of $\langle S, T \rangle$ -ergodic measures also experiences a *complexity gap*, in a sense we make precise. We show (Theorem 1.9) that if μ has *low complexity* (meaning that a certain function grows subquadratically), then it actually has *bounded complexity* (meaning that this function is bounded) and μ is atomic. Moreover, all atomic measures have bounded complexity.

1.3. Rephrasing $\times p, \times q$ in symbolic terms. We begin by recasting Furstenberg's Conjecture and the Rudolph-Johnson Theorem as statements about symbolic dynamical systems. We start by setting some terminology and notation.

A (*measure preserving*) *system* (X, \mathcal{X}, μ, G) is a measure space X with an associated σ -algebra \mathcal{X} , probability measure μ , and an abelian group G of measurable, measure preserving transformations. If the context is clear, we omit the σ -algebra from the notation, writing (X, μ, G) , and call it a *system*. The system (X, μ, G) is *free* if the set $\{x \in X : gx = x\}$ has measure 0 for every $g \in G$ and the system is *ergodic* if the only sets invariant under the action of G have either trivial or full measure. It follows that if (X, μ, G) is an ergodic system with an abelian group G of transformations, then the action of G is free if $g_1^{n_1} \circ \dots \circ g_k^{n_k} \neq \text{Id}$ for any $g_1, \dots, g_k \in G$ and $(n_1, \dots, n_k) \neq (0, \dots, 0)$.

Two systems $(X_1, \mathcal{X}_1, \mu_1, G)$ and $(X_2, \mathcal{X}_2, \mu_2, G)$ are (*measure theoretically*) *isomorphic* if there exist $X'_1 \in \mathcal{X}_1$ and $X'_2 \in \mathcal{X}_2$ with $\mu_1(X'_1) = \mu_2(X'_2) = 1$ such that $gX'_1 \subset X'_1$ for all $g \in G$ and $gX'_2 \subset X'_2$ for all $g \in G$, and there is an invertible bimeasurable transformation $\pi : X'_1 \rightarrow X'_2$ such that $\pi_*\mu_1 = \mu_2$ and $\pi g(x) = g\pi(x)$ for all $x \in X'_1, g \in G$.

We are particularly interested in the \mathbb{Z}^2 -system generated by the two commuting, measure preserving transformations S and T . In this case, we write $(X, \mathcal{X}, \mu, S, T)$ for the \mathbb{Z}^2 -system.

A (*topological*) *system* (X, G) , is a compact metric space X and a group G of homeomorphisms mapping X to itself. If it is clear from the context that we are referring to a topological system, we call (X, G) a *system*. A system is said to be *minimal* if for any $x \in X$, the orbit $\{gx : g \in G\}$ is dense in X . By the Krylov-Bogolyubov Theorem, every system (X, G) admits an invariant Borel probability measure and if this measure is unique, we say that (X, G) is *uniquely ergodic*. A system (X, G) is *strictly ergodic* if it is both minimal and uniquely ergodic.

Let \mathcal{A} denote a finite alphabet and let $\mathcal{A}^{\mathbb{Z}^2}$ be the set of \mathcal{A} -colorings of \mathbb{Z}^2 . For $x \in \mathcal{A}^{\mathbb{Z}^2}$ and $\vec{u} \in \mathbb{Z}^2$, we denote the element of \mathcal{A} that x assigns to \vec{u} by $x(\vec{u})$. With respect to the metric

$$d(x, y) := 2^{-\inf\{|\vec{u}| : x(\vec{u}) \neq y(\vec{u})\}},$$

$\mathcal{A}^{\mathbb{Z}^2}$ is compact and the leftward and downward shift maps $\sigma, \tau: \mathcal{A}^{\mathbb{Z}^2} \rightarrow \mathcal{A}^{\mathbb{Z}^2}$ given by

$$(\sigma x)(i, j) := x(i + 1, j), \quad (1)$$

$$(\tau x)(i, j) := x(i, j + 1) \quad (2)$$

are homeomorphisms. A closed set $X \subset \mathcal{A}^{\mathbb{Z}^2}$ which is invariant under the joint action of $\langle \sigma, \tau \rangle$ is called a \mathbb{Z}^2 -subshift. (The analogous definitions hold for \mathbb{Z}^d -subshifts.)

A uniquely ergodic topological system (\hat{X}, ν, G) is said to be a *topological model* for the measure preserving system (X, \mathcal{X}, μ, G) if there exists a measure theoretic isomorphism between (\hat{X}, ν, G) and (X, μ, G) . Again, we are mainly interested in topological systems generated by two transformations, and in this case we denote the topological system by (\hat{X}, σ, τ) .

The Jewett-Krieger Theorem [5, 8] states that any ergodic \mathbb{Z} -system has a strictly ergodic topological model, meaning that the system is measure theoretically isomorphic to a minimal, uniquely ergodic topological system. This was generalized to cover ergodic \mathbb{Z}^d -systems by Weiss [18], and further refined by Rosenthal (we only state it for \mathbb{Z}^2 , as this is the only case relevant for our purposes):

Theorem 1.3 (Rosenthal [15]). *Let $(X, \mathcal{X}, \mu, S, T)$ be an ergodic, free \mathbb{Z}^2 -system with entropy less than $\log k$. Then there exists a minimal, uniquely ergodic subshift $\hat{X} \subset \{1, \dots, k\}^{\mathbb{Z}^2}$ such that if $\sigma, \tau: \hat{X} \rightarrow \hat{X}$ denote the horizontal and vertical shifts (respectively) and if ν is the unique invariant Borel probability on \hat{X} and \mathcal{B} denotes the Borel σ -algebra, then $(\hat{X}, \mathcal{B}, \nu, \sigma, \tau)$ is a topological model for $(X, \mathcal{X}, \mu, S, T)$.*

We note that in [15], the proof given shows that $\hat{X} \subset \{1, \dots, k + 1\}^{\mathbb{Z}^2}$ and the result that the shift alphabet can be taken to have only k letters is stated without proof. However, the size of the alphabet is not relevant for our purposes, other than the fact that it is a finite number.

The subshift $\hat{X} \subset \{1, \dots, k\}^{\mathbb{Z}^2}$ in the conclusion of Theorem 1.3 is not uniquely defined, and so we make the following definition:

Definition 1.4. Let $(X, \mathcal{X}, \mu, S, T)$ be an ergodic \mathbb{Z}^2 -system. A minimal, uniquely ergodic \mathbb{Z}^2 -subshift that is measure theoretically isomorphic to $(X, \mathcal{X}, \mu, S, T)$ is called a *Jewett-Krieger model* for $(X, \mathcal{X}, \mu, S, T)$.

Theorem 1.3 guarantees that any free ergodic \mathbb{Z}^2 system of finite entropy has a Jewett-Krieger model. However the definition is still valid for non-free, ergodic \mathbb{Z}^2 systems; the only difference is that Rosenthal's Theorem no longer guarantees that such a model exists. For the case of interest to us, we show (in the proof of Theorem 1.9) that if μ is $\langle S, T \rangle$ -ergodic, then either μ is atomic or the action of $\langle S, T \rangle$ is free. This motivates us to make the following observation: a finite, ergodic \mathbb{Z}^2 -system cannot be free, but it has a Jewett-Krieger model in a trivial way, obtained by partitioning the system into individual points.

Using this language, we can rephrase Furstenberg's Conjecture and the Rudolph-Johnson Theorem as equivalent statements about Jewett-Krieger models. Fix the transformations $S, T: [0, 1) \rightarrow [0, 1)$ to be the maps $Sx := px \pmod{1}$ and $Tx := qx \pmod{1}$, where $p, q \geq 1$ are multiplicatively independent integers. By the natural extension, we mean the invertible cover (see Section 2.1).

Conjecture 1.5 (Symbolic Furstenberg Conjecture). *Let μ be a Borel probability measure on $[0, 1)$ with Borel σ -algebra \mathcal{B} that is invariant under both S and T and ergodic for the joint action. If $\hat{X} \subset \{0, 1\}^{\mathbb{Z}^2}$ is a Jewett-Krieger model for the natural extension of $([0, 1), \mathcal{B}, \mu, S, T)$, then either \hat{X} is finite or μ is Lebesgue measure.*

Theorem 1.6 (Symbolic Rudolph-Johnson Theorem). *Let μ be a Borel probability measure on $[0, 1)$ with Borel σ -algebra \mathcal{B} which is invariant under both S and T and is ergodic for the joint action. Let $\hat{X} \subset \{0, 1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0, 1), \mathcal{B}, \mu, S, T)$ and let $\sigma, \tau: \hat{X} \rightarrow \hat{X}$ denote the horizontal and vertical shifts (respectively). If either $h_\nu(\sigma) > 0$ or $h_\nu(\tau) > 0$, then μ is Lebesgue measure.*

Proof. An isomorphism of the \mathbb{Z}^2 -systems $(X, \mathcal{X}, \mu, S, T)$ and $(\hat{X}, \mathcal{B}, \nu, \sigma, \tau)$ restricts to an isomorphism of the \mathbb{Z} -systems (X, \mathcal{X}, μ, S) and $(\hat{X}, \mathcal{B}, \nu, \sigma)$, and so $h_\mu(S) = h_\nu(\sigma)$. Similarly $h_\mu(T) = h_\nu(\tau)$. The statement then follows immediately from the Rudolph-Johnson Theorem. \square

1.4. Combinatorial rephrasing of measure theoretic entropy. The appeal of Theorem 1.6 is that the hypothesis that $h_\nu(\sigma) > 0$ (or equivalently that $h_\nu(\tau) > 0$) can be phrased purely as a combinatorial statement about the frequency with which words in the language of \hat{X} occur in larger words in the language of \hat{X} . To explain this, we start with some definitions.

If $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift over the finite alphabet \mathcal{A} , we write $x = (x(n): n \in \mathbb{Z})$. A *word* is defined to be a finite sequence of symbols contained consecutively in some x and we let $|w|$ denote the number of symbols in w (it may be finite or infinite). A word w is a *subword* of a word u if the symbols in the word w occur somewhere in u as consecutive symbols. The *language* $\mathcal{L} = \mathcal{L}(X)$ of X is defined to be the collection of all finite subwords that arise in elements of X . If $w \in \mathcal{L}(X)$, let $[w]$ denote the *cylinder set* it determines, meaning that

$$[w] = \{u \in \mathcal{L}: u(n) = w(n) \text{ for } 1 \leq n \leq |w|\}.$$

These definitions naturally generalize to a two dimensional subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$, and for $x \in \mathcal{A}^{\mathbb{Z}^2}$ we write $x = (x(\vec{u}): \vec{u} \in \mathbb{Z}^2)$. A *word* is a finite, two dimensional configuration that is convex and connected (as a subset of \mathbb{Z}^2), and a *subword* is a configuration contained in another word. If $F \subset \mathbb{Z}^2$ is finite and $\beta \in \mathcal{A}^F$, then the *cylinder set of shape F determined by β* is defined to be the set

$$[F; \beta] := \{x \in \mathcal{A}^{\mathbb{Z}^2}: x(\vec{u}) = \beta(\vec{u}) \text{ for all } \vec{u} \in F\}.$$

Lemma 1.7. *Let $(\hat{X}, \mathcal{B}, \nu, \sigma, \tau)$ be a strictly ergodic \mathbb{Z}^2 -subshift. Let w be a $(2n+1) \times (2n+1)$ word in the language of \hat{X} and let $[w]$ denote the cylinder set determined by placing the word w centered at $(0, 0)$. Let u_1, u_2, u_3, \dots be words in the language of \hat{X} such that u_i is a square of size $(2n+2i+1) \times (2n+2i+1)$. If $N(w, u_i)$ denotes the number of times w occurs as a subword of u_i , then*

$$\nu[w] = \lim_{i \rightarrow \infty} N(w, u_i) / (2i+1)^2.$$

Proof. By unique ergodicity, the Birkhoff averages of a continuous function converge uniformly to the integral of the function. In particular, this applies to the continuous function $1_{[w]}$, so the limit exists and is independent of the sequence $\{u_i\}_{i=1}^\infty$. \square

For $m, n \in \mathbb{N}$, let $\mathcal{P}(m, n)$ be the partition of \widehat{X} according to cylinder sets of shape $[0, m-1] \times [-n+1, n-1]$. Observe that (recall that σ , as defined in (1), denotes the left shift)

$$\mathcal{P}(m, n) = \bigvee_{i=0}^k \sigma^{-i} \mathcal{P}(1, n)$$

and that $\bigvee_{i=-k}^k \sigma^i \mathcal{P}(1, n)$ is the partition of \widehat{X} into symmetric $(2m+1) \times (2n+1)$ -cylinders centered at the origin. Therefore, $\{\mathcal{P}(1, n)\}_{n=1}^\infty$ is a sufficient (in the sense of Definition 4.3.11 in [7]) family of partitions to generate the Borel σ -algebra of the system $(\widehat{X}, \mathcal{B}, \nu, \sigma)$, where we view this as a \mathbb{Z} -system with respect to the horizontal shift σ . Let $h_\nu(\sigma, \mathcal{Q})$ denote the measure theoretic entropy of the system $(\widehat{X}, \mathcal{B}, \nu, \sigma)$ with respect to the partition \mathcal{Q} and let $h_\nu(\sigma)$ denote the measure theoretic entropy of the system. It follows that

$$\begin{aligned} h_\nu(\sigma) &= \sup_n h_\nu(\sigma, \mathcal{P}(1, n)) \\ &= \lim_{n \rightarrow \infty} h_\nu(\sigma, \mathcal{P}(1, n)) \\ &= - \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{w \in \mathcal{P}(m, n)} \nu[w] \log \nu[w] \\ &= - \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \frac{1}{m} \sum_{w \in \mathcal{P}(m, n)} \frac{N(w, u_i)}{(2i+1)^2} \cdot \log \frac{N(w, u_i)}{(2i+1)^2} \end{aligned}$$

by Lemma 1.7. In other words, the Rudolph-Johnson Theorem is equivalent to:

Theorem 1.8 (Combinatorial Rudolph-Johnson Theorem). *Let μ be a Borel probability measure on $[0, 1)$ with Borel σ -algebra \mathcal{B} and assume that μ is invariant under both S and T , and ergodic for the joint action. Let \widehat{X} be a Jewett-Krieger model for the natural extension of $([0, 1), \mathcal{B}, \mu, S, T)$ and without loss suppose the horizontal shift on \widehat{X} is intertwined with S under this isomorphism. If*

$$- \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} \frac{1}{m} \sum_{w \in \mathcal{P}(m, n)} \frac{N(w, u_i)}{(2i+1)^2} \cdot \log \frac{N(w, u_i)}{(2i+1)^2} > 0,$$

then the value of this limit is $\log p$ and μ is Lebesgue measure.

1.5. Complexity of subshifts. If $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a nonempty subshift, then its *complexity function* is the function $P_X: \{\text{finite subsets of } \mathbb{Z}^2\} \rightarrow \mathbb{N}$ given by

$$P_X(F) := |\{\beta \in \mathcal{A}^F: [F; \beta] \cap X \neq \emptyset\}|.$$

Let $R_n := \{(i, j) \in \mathbb{Z}^2: 1 \leq i, j \leq n\}$ denote the $n \times n$ rectangle in \mathbb{Z}^2 . A standard notion of the complexity of a subshift $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is the asymptotic growth rate of $P_X(R_n)$. Observe that $P_X(R_n)$ is bounded (in n) if and only if X is finite. Moreover, $P_X(R_n)$ grows exponentially if and only if (X, σ, τ) has positive topological entropy.

We are now in a position to state our main technical result.

Theorem 1.9. *Let μ be a Borel probability measure on $[0, 1)$ with Borel σ -algebra \mathcal{B} . Assume that μ is invariant under both S and T and ergodic for the joint action, and let $\widehat{X} \subset \{0, 1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0, 1), \mathcal{B}, \mu, S, T)$. If there exists $n \in \mathbb{N}$ such that $P_{\widehat{X}}(R_n) \leq \frac{1}{2}n^2$, then $P_{\widehat{X}}(R_n)$ is bounded (independent of n) and \widehat{X} is finite. In particular, μ is atomic.*

This gives a nontrivial complexity gap for the set of $\langle S, T \rangle$ -ergodic probability measures, which is our main result:

Corollary 1.10 (Complexity gap). *Let μ be a Borel probability measure on $[0, 1]$ which is invariant under both S and T and ergodic for the joint action, and let $\widehat{X} \subset \{0, 1\}^{\mathbb{Z}^2}$ be a Jewett-Krieger model for the natural extension of $([0, 1], \mathcal{B}, \mu, S, T)$. Then either $P_{\widehat{X}}(R_n)$ is bounded (and μ is atomic) or*

$$\liminf_{n \rightarrow \infty} \frac{P_{\widehat{X}}(R_n)}{n^2} \geq \frac{1}{2}.$$

This gap is nontrivial in the following sense: there exist aperiodic, strictly ergodic \mathbb{Z}^2 -subshifts whose complexity function grows subquadratically. The statement made by Corollary 1.10 is that any such subshift cannot be a Jewett-Krieger model of any $\langle \times p, \times q \rangle$ -ergodic measure on $[0, 1]$.

Example 1.11. Let $X \subset \{0, 1\}^{\mathbb{Z}}$ be a Sturmian shift (see [10] for the definition). Then X is strictly ergodic and $P_X(n) = n + 1$ for all $n \in \mathbb{N}$. Let $Y \subset \{0, 1\}^{\mathbb{Z}^2}$ be the subshift whose points are obtained by placing each $x \in X$ along the x -axis in \mathbb{Z}^2 and then copying vertically (i.e. each point in Y is vertically constant and its restriction to the x -axis is an element of X). It follows that Y is strictly ergodic and that $P_Y(R_n) = n + 1$ for all $n \in \mathbb{N}$. Corollary 1.10 shows that Y is not a Jewett-Krieger model for any $\langle \times p, \times q \rangle$ -ergodic measure on $[0, 1]$.

1.6. Remarks on complexity growth. We conclude our introduction with a few brief remarks on Theorem 1.9 and Corollary 1.10. We show (Lemma 2.1) that any Jewett-Krieger model \widehat{X} for an atomic $\langle S, T \rangle$ -ergodic measure is a strictly ergodic \mathbb{Z}^2 -subshift containing only doubly periodic \mathbb{Z}^2 -colorings, meaning that there are only finitely many points in \widehat{X} . From this, it is easy to deduce that $P_{\widehat{X}}(R_n)$ is bounded independently of n (by the number of points in \widehat{X}). Moreover, we show that if \widehat{X} is a Jewett-Krieger model for μ and if \widehat{X} contains only doubly periodic \mathbb{Z}^2 -colorings, then μ is atomic.

A strategy for proving Theorem 1.9 is therefore to find a nontrivial growth rate of $P_{\widehat{X}}(R_n)$ which implies that \widehat{X} contains only doubly periodic \mathbb{Z}^2 -colorings. A simple example of such a rate follows from the classical Morse-Hedlund Theorem [10]: if there exists $n \in \mathbb{N}$ such that $P_{\widehat{X}}(R_n) \leq n$, then \widehat{X} contains only doubly periodic \mathbb{Z}^2 -colorings (see e.g. the proof of Theorem 1.2 in [13]). In fact this bound is sharp: there exist \mathbb{Z}^2 -colorings that are not doubly periodic and yet satisfy $P_{\widehat{X}}(R_n) = n + 1$ for all $n \in \mathbb{N}$. Many other subquadratic growth rates can also be realized by strictly ergodic \mathbb{Z}^2 -subshifts that do not contain doubly periodic points (see, for example, [12]). Therefore, a weak version of Theorem 1.9 that replaces the assumption that there exists $n \in \mathbb{N}$ such that $P_{\widehat{X}}(R_n) \leq \frac{1}{2} \cdot n^2$ with the stronger assumption that there exists $n \in \mathbb{N}$ such that $P_{\widehat{X}}(R_n) \leq n$, follows from the Morse-Hedlund Theorem. However, this weak theorem relies on the fact that there are simply no strictly ergodic \mathbb{Z}^2 -subshifts for which $P_{\widehat{X}}(R_n)$ is unbounded but for which $P_{\widehat{X}}(R_n) \leq n$ (for some n). The complexity gap provided by this weak theorem is therefore trivial in the sense that there are no strictly ergodic \mathbb{Z}^2 -subshifts whose complexity function lies in this gap.

On the other hand, there do exist strictly ergodic \mathbb{Z}^2 -subshifts with unbounded complexity and such that $P_{\widehat{X}}(R_n) < \frac{1}{2} \cdot n^2$. This is the interest in Theorem 1.9 and Corollary 1.10. The content of the theorem is that although such \mathbb{Z}^2 -systems

exist, they can not be Jewett-Krieger models of $\langle S, T \rangle$ -ergodic measures on $[0, 1]$. This is analogous to Theorem 1.8, which says that although there are strictly ergodic \mathbb{Z}^2 -subshifts that have small but positive entropy, they are not Jewett-Krieger models of $\langle S, T \rangle$ -ergodic measures on $[0, 1]$. Moreover, analogous to the hypothesis of Corollary 1.10 which relies on the growth rate of $P_{\hat{X}}(\cdot)$, the hypothesis of Theorem 1.8 is a condition on the growth rate of the relative complexity function $N(\cdot, \cdot)$ of Lemma 1.7, with respect to the action of the horizontal shift (a similar statement holds for the vertical shift).

2. PROOF OF THEOREM 1.9

Throughout this section, we assume that $p, q \geq 2$ are multiplicatively independent integers and that μ is a Borel probability measure on $[0, 1]$ which is invariant under both

$$\begin{aligned} Sx &:= px \pmod{1}; \\ Tx &:= qx \pmod{1} \end{aligned}$$

and is ergodic with respect to the joint action $\langle S, T \rangle$. Let \mathcal{B} denote the associated Borel σ -algebra on $[0, 1]$

2.1. The natural extension. Let X be the natural extension of the \mathbb{N}^2 -system $([0, 1], \mathcal{B}, \mu, S, T)$. Specifically (following [14]), let

$$X := \left\{ y \in [0, 1]^{\mathbb{Z}^2} : y(i+1, j) = Sy(i, j) \text{ and } y(i, j+1) = Ty(i, j) \text{ for all } i, j \in \mathbb{Z} \right\},$$

and for $(i, j) \in \mathbb{Z}^2$ let $\pi_{(i, j)} : X \rightarrow [0, 1]$ be the map $\pi_{(i, j)}(y) = y(i, j)$. Define a countably additive measure μ_X on the σ -algebra

$$\bigcup_{i=0}^{\infty} \pi_{(-i, -i)}^{-1} \mathcal{B}$$

by setting $\mu_X(\pi_{(-i, -i)}^{-1} A) := \mu(A)$. Let \mathcal{X} be the completion of this σ -algebra with respect to μ_X . Let $S_X, T_X : X \rightarrow X$ be the left shift and the down shift, respectively. Thus $\pi_{(0, 0)}$ defines a measure theoretic factor map from $(X, \mathcal{X}, \mu_X, S_X, T_X)$ to $([0, 1], \mathcal{B}, \mu, S, T)$. Moreover, μ_X is ergodic if and only if μ is ergodic. By construction, $h_\mu(S) = h_{\mu_X}(S_X)$, $h_\mu(T) = h_{\mu_X}(T_X)$, and $h_\mu(\langle S, T \rangle) = h_{\mu_X}(\langle S_X, T_X \rangle)$.

The advantage of working with $(X, \mathcal{X}, \mu_X, S_X, T_X)$ instead of the original system is that the natural extension is an ergodic \mathbb{Z}^2 -system.

2.2. Jewett-Krieger models and periodicity. If the two dimensional entropy of a system is positive, then the entropy of every one dimensional subsystem is infinite (for a proof, see, for example, [17]). In our setting, since $h_\mu(S) \leq h_{\text{top}}(S) = \log(p)$ (and $h_\mu(T) \leq h_{\text{top}}(T) = \log(q)$), it follows that the measure theoretic entropy of the joint action generated by $\langle S, T \rangle$ on $[0, 1]$ with respect to μ is also zero. It follows that the measure theoretic entropy with respect to μ_X of the joint action on X generated by $\langle S_X, T_X \rangle$ is zero. Therefore, by Theorem 1.3, there exists a strictly ergodic subshift $\hat{X} \subset \{0, 1\}^{\mathbb{Z}^2}$ such that $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is measure theoretically isomorphic to $(\hat{X}, \hat{\mathcal{X}}, \nu, \sigma, \tau)$, where $\hat{\mathcal{X}}$ is the Borel σ -algebra on \hat{X} , $\sigma, \tau : \hat{X} \rightarrow \hat{X}$ denote the left shift and down shift (respectively), and ν is the unique $\langle \sigma, \tau \rangle$ -invariant Borel probability measure. Note that the choice of \hat{X} is not necessarily unique.

Lemma 2.1. *If $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is an atomic system, then any Jewett-Krieger model $(\widehat{X}, \widehat{\mathcal{X}}, \nu, \sigma, \tau)$ for $(X, \mathcal{X}, \mu_X, S_X, T_X)$ is finite.*

Proof. Let $\pi: (\widehat{X}, \widehat{\mathcal{X}}, \nu, \sigma, \tau) \rightarrow (X, \mathcal{X}, \mu_X, S_X, T_X)$ be an isomorphism and let $x \in X$ be an atom. Then there exist full measure sets $\widehat{X}_1 \subset \widehat{X}$ and $X_1 \subset X$ such that $\pi: \widehat{X}_1 \rightarrow X_1$ is a bijection which intertwines the \mathbb{Z}^2 actions. Every atom in X is contained in X_1 , and if $x \in X_1$ is an atom then there exists unique $y \in \widehat{X}_1$ such that $\pi(y) = x$. It follows that $\nu(\{y\}) = \mu_X(\{x\}) > 0$ and so y is an atom in \widehat{X} . By the Poincaré Recurrence Theorem, there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $S_X^i T_X^j y = y$. Let $V_y := \{(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\} : S_X^i T_X^j y = y\}$ be the (nonempty) set of nontrivial period vectors for y . If $\dim(\text{Span}(V_y)) = 1$, then

$$\lim_{N \rightarrow \infty} \frac{1}{(2N+1)^2} \sum_{-N \leq i, j \leq N} 1_{\{y\}}(S_X^i T_X^j y) = 0 < \nu(\{y\}),$$

which contradicts the pointwise ergodic theorem. Therefore $\dim(\text{Span}(V_y)) = 2$ and $y \in \mathcal{A}^{\mathbb{Z}^2}$ is doubly periodic. Moreover, for ν -a.e. $z \in \widehat{X}$ we have $S_X^i T_X^j z = y$ for some $(i, j) \in \mathbb{Z}^2$ and so z is also doubly periodic (with periods equal to those of y). Thus there are only finitely many points $z \in \widehat{X}$. \square

Since \widehat{X} is minimal, and hence transitive, we can use the following tool for studying the dynamics of $(X, \mathcal{X}, \mu_X, S_X, T_X)$:

Theorem 2.2 (Cyr & Kra [2]). *If (X, σ, τ) is a transitive \mathbb{Z}^2 -subshift and there exist $n, k \in \mathbb{N}$ such that $P_X(n, k) \leq nk/2$, then there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $\sigma^i \tau^j x = x$ for all $x \in X$.*

Lemma 2.3. *If there exists $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $\sigma^i \tau^j x = x$ for every $x \in \widehat{X}$, then $S_X^i T_X^j x = x$ for μ -almost every $x \in X$.*

Proof. Let $\psi: \widehat{X} \rightarrow X$ be an isomorphism. Thus there exist $\widehat{X}_1 \subset \widehat{X}$ and $X_1 \subset X$ such that $\nu(\widehat{X}_1) = \mu_X(X_1) = 1$, $\psi: \widehat{X}_1 \rightarrow X_1$ is a bi-measurable bijection, $\psi_* \nu = \mu_X$, $\psi \circ \sigma = S_X \circ \psi$, and $\psi \circ \tau = T_X \circ \psi$. Let $E = \{x \in X_1 : S_X^i T_X^j x \neq x\}$. Since $\psi^{-1}(E) = \{y \in \widehat{X}_1 : \sigma^i \tau^j y \neq y\}$, it follows that $\mu_X(E) = \nu(\psi^{-1}(E)) = 0$. \square

Theorem 2.4. *If there exist $n, k \in \mathbb{N}$ such that $P_{\widehat{X}}(n, k) \leq nk/2$, then μ is atomic. Moreover, if \widehat{Y} is any other Jewett-Krieger model for $([0, 1], \mathcal{B}, \mu, S, T)$, then $P_{\widehat{Y}}(n, k)$ is bounded independent of $n, k \in \mathbb{N}$.*

Proof. Combining Theorem 2.2 and Lemma 2.3, there exist $(i, j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $S_X^i T_X^j x = x$ for μ_X -a.e. $x \in X$. Therefore $(S_X^i T_X^j x)(0, 0) = x(0, 0)$ for μ_X -a.e. $x \in X$. It is immediate that we also have $(S_X^{-i} T_X^{-j} x)(0, 0) = x(0, 0)$ for μ_X -a.e. $x \in X$. So there are two cases to consider, depending on the sign of $i \cdot j$.

Case 1. Suppose $i \cdot j \geq 0$. Then, replacing by $-i$ and $-j$ if necessary, we can assume that both i and j are nonnegative. Set $E := \{y \in [0, 1] : S^i T^j y \neq y\}$ and let $y \in E$. Then if $x \in \pi^{-1}(y)$, we have that $S_X^i T_X^j x \neq x$. Thus $\mu(E) = \mu_X(\pi^{-1}(E)) = 0$ and so $S^i T^j y = y$ for μ -a.e. $y \in [0, 1]$.

Now observe that $S^i T^j y = y$ is equivalent to the statement that

$$p^i q^j y = y \pmod{1},$$

which only has finitely many solutions in the interval $[0, 1)$. Therefore, μ is supported on a finite set. Since μ is $\langle S, T \rangle$ -invariant, this set must be S - and T -invariant. Therefore there exist $a, b \in \mathbb{N}$ such that S^a and T^b are both equal to the identity μ -almost everywhere.

Case 2. Suppose $i \cdot j < 0$. Again, replacing by $-i$ and $-j$ if necessary, we can assume that $i < 0$ and $j > 0$. Now set $E := \{y \in [0, 1) : S^{|i|}y \neq T^jy\}$. Thus if $y \in E$ and $x \in \pi^{-1}(y)$, then $x(-i, j) \neq x(0, 0) = y$ as $S^{|i|}(x(-i, j)) = x(0, j) = T^j(x(0, 0))$ by construction. Therefore $S^i T^j x \neq x$ and so $\mu(E) = \mu_X(\pi^{-1}(E)) = 0$. It follows that $S^{|i|}y = T^jy$ for μ -a.e. $y \in [0, 1)$.

Finally observe that $S^{|i|}y = T^jy$ is equivalent to

$$p^{|i|}y = q^jy \pmod{1}.$$

As p and q are multiplicatively independent, there are only finitely many solutions in the interval $[0, 1)$. Therefore, again, μ is supported on a finite set and there exist $a, b \in \mathbb{N}$ such that S^a and T^b are both equal to the identity μ -almost everywhere.

This establishes the first claim of the theorem. By Lemma 2.1, any Jewett-Krieger model of an atomic system is finite, and the second statement follows. \square

We use this to complete the proof of Theorem 1.9:

Proof of Theorem 1.9. Let μ be a Borel probability measure on $[0, 1)$ that is $\langle S, T \rangle$ -ergodic. If this two dimensional action is not free, arguing as in the proof of Theorem 2.4 that μ is an atomic measure, we are done. Thus we can assume that the action is free, and similarly the action for the natural extension is also free.

Let $(\widehat{X}, \widehat{\mathcal{X}}, \nu, \sigma, \tau)$ be a Jewett-Krieger model for the natural extension of the system $([0, 1), \mathcal{B}, \mu, S, T)$. If there is no such model satisfying the additional property that there exist $n, k \in \mathbb{N}$ satisfying $P_{\widehat{X}}(n, k) \leq nk/2$, then the conclusion of the Theorem holds vacuously. Thus it suffices to assume that there exists a Jewett-Krieger model $(\widehat{X}, \widehat{\mathcal{X}}, \nu, \sigma, \tau)$ with the property that there exist $n, k \in \mathbb{N}$ satisfying $P_{\widehat{X}}(n, k) \leq nk/2$. By Theorem 2.4, $([0, 1), \mathcal{B}, \mu, S, T)$ is atomic. \square

3. HIGHER DIMENSIONS

Theorem 1.9 shows that if μ is any nonatomic $\times p, \times q$ ergodic measure then the natural extension of $([0, 1), \mathcal{X}, \mu, S, T)$ cannot be measurably isomorphic to any \mathbb{Z}^2 -subshift of whose complexity function satisfies $P_X(n, n) = o(n^2)$. It is natural to ask whether this result can be generalized to higher dimensions. In particular, if p_1, \dots, p_d are a multiplicatively independent set of integers and μ is a nonatomic $\times p_1, \dots, \times p_d$ ergodic measure, we can ask if the natural extension of $(X, \mathcal{X}, \mu, \times p_1, \dots, \times p_d)$ could have a topological model whose complexity function is $o(n^d)$.

The same method used in the two dimensional case suggests a path to proving this result. If one could show that any free, strictly ergodic \mathbb{Z}^d -subshift whose complexity function is $o(n^d)$ is periodic, then it would follow that no such topological model for μ exists. However, the analog of Theorem 2.2 in dimension $d > 2$ is false. Julien Cassaigne [1] has shown that for $d > 2$, there exists a minimal \mathbb{Z}^d -subshift X whose elements are not periodic in any direction, and is such that for any $\varepsilon > 0$ we have $P_X(n, n, \dots, n) = o(n^{2+\varepsilon})$. On the other hand, the authors have recently shown [3] that the analog of Theorem 2.2 does hold for dimension $d > 2$ if a certain expansiveness assumption is imposed on the subshift.

If $Y \subset \mathcal{A}^{\mathbb{Z}^d}$ is a subshift, then we say that the x -axis in \mathbb{Z}^d is *strongly expansive* if whenever $x, y \in X$ have the same restriction to the x -axis, we have $x = y$. In this case, if $X \subset \mathcal{A}^{\mathbb{Z}}$ is the subshift obtained by restricting elements of Y to the x -axis, then there exist homeomorphisms $\tau_1, \dots, \tau_{d-1}: X \rightarrow X$ which commute pairwise and with the shift σ and are such that for any $y \in Y$ we have $y(i_1, i_2, \dots, i_d) = \left(\tau_1^{i_1} \tau_2^{i_2} \dots \tau_{d-1}^{i_{d-1}} \sigma^{i_d} \pi_X(y) \right) (0)$ for all $i_1, \dots, i_d \in \mathbb{Z}^d$, where $\pi_X(y)$ denotes the restriction of y to the x -axis. In previous work, we have shown that:

Theorem 3.1 (Cyr & Kra [3]). *Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a minimal subshift and let $\tau_1, \dots, \tau_{d-1}: X \rightarrow X$ be homeomorphisms of X that commute with the shift σ . If $\langle \sigma, \tau_1, \dots, \tau_{d-1} \rangle \cong \mathbb{Z}^d$, then $\liminf_{n \rightarrow \infty} P_X(n)/n^d > 0$.*

With some additional effort, the same result can be shown if the assumption that (X, σ) is minimal (as a \mathbb{Z} -system) is relaxed to only require that $(X, \sigma, \tau_1, \dots, \tau_d)$ is minimal (as a \mathbb{Z}^d -system). Thus, the only obstruction to generalizing Theorem 1.9 to the higher dimensional setting is the following:

Conjecture 3.2. *For every nonatomic Borel probability μ on $[0, 1)$ which is ergodic for the joint action of $\times p_1, \dots, \times p_d$, there is a strongly expansive, minimal topological model for $(X, \mathcal{X}, \mu, \times p_1, \dots, \times p_d)$.*

If this conjecture holds, then it follows that any such system is measurably isomorphic to a subshift whose complexity function grows on the order of n^d .

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